Stability of Planar Nonlinear Switched Systems

Ugo Boscain,
SISSA-ISAS, Via Beirut 2-4, 34014 Trieste, Italy. E-mail: boscain@sissa.it
Grégoire Charlot
Université Montpellier II, Mathématiques, CC051, 34095 Montpellier Cedex 5, France. E-mail: cha@math.univ-montp2.fr
Mario Sigalotti
INRIA, 2004 route des lucioles, 06902 Sophia Antipolis, France. E-mail: Mario.Sigalotti@sophia.inria.fr

***Abstract* — We consider the time-dependent nonlinear system $\dot{q}(t) = u(t)X(q(t)) + (1 - u(t))Y(q(t))$, where $q \in \mathbb{R}^2$, X and Y are two smooth vector fields, globally asymptotically stable at the origin and $u : [0,\infty) \to \{0,1\}$ is an arbitrary measurable function. Analysing the topology of the set where X and Y are parallel, we give some sufficient and some necessary conditions for global asymptotic stability, uniform with respect to $u(.)$. Such conditions can be verified without any integration or construction of a Lyapunov function, and they are robust under small perturbations of the vector fields.**

Keywords — Global asymptotic stability, planar switched systems, nonlinear.

1 Introduction

A *switched system* is a family of continuous-time dynamical systems endowed with a rule that determines, at every time, which dynamical system is responsible for the time evolution. More precisely let $\{f_u \mid u \in U\}$ be a (possibly infinite) set of smooth vector fields on a manifold M, and consider, as u varies in U, the family of dynamical systems

$$\dot{q} = f_u(q)\,, \qquad q \in M\,. \tag{1}$$

A non-autonomous dynamical system is obtained by assigning a so-called *switching function* $u(.) : [0,\infty) \to U$.

In this paper, the switching function models the behavior of a parameter which cannot be predicted a priori. It represents some phenomena (e.g., a disturbance) that it is not possible to control or include in the dynamical system model.

A typical problem related to switched systems is to obtain, out of a property which is shared by all the autonomous dynamical systems governed by the vector fields f_u, some, maybe weaker, property for the time-dependent system associated with an arbitrary switching function $u(.)$. For a discussion on various issues related to switched systems we refer the reader to [11, 13].

In this paper, we consider a two-dimensional nonlinear switched system of the type

$$\dot{q} = u\,X(q) + (1-u)\,Y(q)\,, \qquad q \in \mathbb{R}^2\,, \qquad u \in \{0,1\}\,, \tag{2}$$

where the two vector fields X and Y are smooth (say, $\mathcal{C}^\infty$) on $\mathbb{R}^2$. In order to define proper non-autonomous systems, we require the switching functions to be measurable.

Assume that $X(0) = Y(0) = 0$ and that the two dynamical systems $\dot{q} = X(q)$ and $\dot{q} = Y(q)$ are globally asymptotically stable at the origin. Our main aim is to study under which conditions on X and Y the origin is globally asymptotically stable for the system (2), uniformly with respect to the switching functions (GUAS for short). For the precise formulation of this and other stability properties, see Definition 1.

In order to study the stability of (2) it is natural to consider its convexification, i.e., the case in which u varies in the whole interval $[0,1]$. It turns out that the stability properties of the two systems are equivalent (see Section 2.2).

The linear version of the system introduced above, namely,

$$\dot{q} = u\,A\,q + (1-u)\,B\,q\,, \quad q \in \mathbb{R}^2\,, \quad u \in \{0,1\}\,, \tag{3}$$

where the 2×2 real matrices A and B have eigenvalues with strictly negative real part, was studied in [6] (see also [14]). More precisely, the results in [6] establish a necessary and sufficient condition for GUAS in terms of three relevant parameters, two depending on the eigenvalues of A and B respectively, and the third one (namely, the cross ratio of the four eigenvectors of A and B in the projective line $\mathbb{C}P^1$) accounting for the interrelations among the two systems. The precise necessary and sufficient condition ensuring GUAS of (3) is quite technical and can be found in [6] (see also [14]). Notice that, in the linear case, GUAS is equivalent to the more often quoted GUES property, i.e., global exponential stability, uniform with respect to the switching rule (see, for example, [3] and references therein). For related results on linear switched systems, see [2, 5, 8, 12, 14].

For nonlinear systems, the problem of characterizing GUAS completely, without assuming the explicit knowledge of the integral curves of X and Y, is hopeless.

The problem, however, admits some partial solution. The purpose of this paper is to provide some sufficient and some necessary conditions for stability which are robust (with respect to small perturbations of the vector fields) and easily verifiable, directly on the vector fields X and Y, without requiring any integration or construction of a Lyapunov function.

Denote by $\mathcal{Z}$ the set on which X and Y are parallel. One of our main results is that, if $\mathcal{Z}$ reduces to the singleton $\{0\}$, then (2) is GUAS (Theorem 6). The proofs works by showing that an admissible trajectory starting from a point $p \in \mathbb{R}^2$ is forced to stay in a compact region bounded by the integral curves of X and Y from p. The fact that X and Y are linearly independent outside the origin plays as a sort of drift which guarantees that the only possible accumulation point of an admissible trajectory is the origin.

When $\mathcal{Z}$ is just compact, we prove that (2) is at least bounded (see Theorem 8). Roughly speaking, this means that its trajectories do not escape to infinity. The idea of the proof is that, if we modify X and Y only in a compact region of the plane, then the boundedness properties of the system are left unchanged. Taking advantage of the result obtained in Theorem 6, we manage to prove the boundedness of (2) by reducing, using compact perturbations, $\mathcal{Z}$ to $\{0\}$, while preserving the global asymptotic stability of X and Y.

Other conditions can be formulated taking into account the relative position of X and Y along $\mathcal{Z}$. Assume that $\mathcal{Z} \setminus \{0\}$ contains at least one point q_0. Since both $X(q_0)$ and $Y(q_0)$ are different from zero, the property of pointing in the same or in the opposite versus can be stated unambiguously. If $X(q_0)$ and $Y(q_0)$ have opposite versus, then there exists a switching function, for the convexified system, whose output is the constant trajectory which stays in q_0. As a consequence, the system (2) is not GUAS.

Additional results can be obtained under the assumption that the pair of vector fields (X, Y) is generic. (For the notion of genericity appropriate to our aims, see Section 2.) In particular, the genericity assumption can be used to guarantee that $\mathcal{Z} \setminus \{0\}$ is an embedded one-dimensional submanifold of the plane. Clearly, $\mathcal{Z}$ needs not to be connected. If the connected component of $\mathcal{Z}$ containing the origin reduces to $\{0\}$ and on all other components X and Y point in the same versus, transversally to $\mathcal{Z}$, then (2) is GUAS. This result is formulated in Theorem 7, which follows the pattern of proof of Theorem 6.

Conversely, Theorem 11 states that, if one connected component of $\mathcal{Z} \setminus \{0\}$ is unbounded and such that X and Y have opposite versus on it, then (2) admits a trajectory going to infinity. Intuitively, this happens because the orientation of $(X(p), Y(p))$ changes while p crosses $\mathcal{Z} \setminus \{0\}$. If $X(p)$ is not tangent to $\mathcal{Z}$ at p and $X(p)$ points in the opposite direction with respect to $Y(p)$, then one can embed $\mathcal{Z}$, locally near p, in a foliation made of admissible trajectories of (2), whose running direction is reversed while crossing $\mathcal{Z}$ (see Figure 1). Since, generically, the points where X is tangent to $\mathcal{Z}$ are isolated, it turns out that there exists an admissible trajectory which tracks globally the unbounded connected component of $\mathcal{Z} \setminus \{0\}$ on which X and Y have opposite versus.

The paper is organized as follows. In Section 2, we recall the main definitions of stability in which we are interested, we introduce the convexified system, and we describe the topological structure of the set $\mathcal{Z}$. The main results are stated in Section 3, where their robustness is also discussed. The proofs are given in Sections 4, 5, 6, and 7.

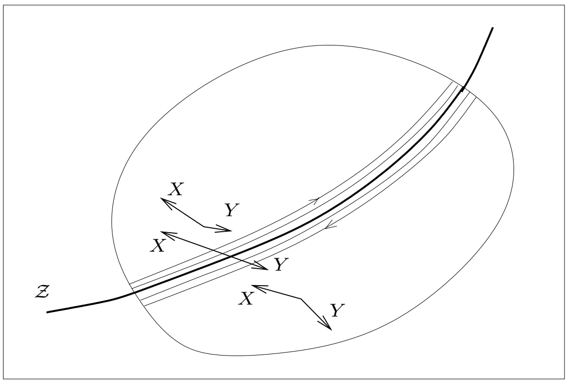

Figure 1: A local foliation embedding $\mathcal{Z}$

2 Basic definitions and facts

2.1 Definitions of stability

Fix $n, m \in \mathbb{N}$ and consider the switched system

$$\dot{q} = f_u(q)\,, \qquad q \in \mathbb{R}^n\,, \qquad u \in U \subset \mathbb{R}^m\,, \tag{4}$$

where U is a measurable subset of $\mathbb{R}^m$ and $(q, u) \mapsto f_u(q)$ is the restriction on $\mathbb{R}^n \times U$ of a $\mathcal{C}^\infty$ function from $\mathbb{R}^n \times \mathbb{R}^m$ to $\mathbb{R}^n$. Assume that $f_u(0) = 0$ for every $u \in U$.
For every $\delta > 0$, denote by $B_\delta \subset \mathbb{R}^n$ the ball of radius δ, centered at the origin. Set

$$\mathcal{U} = \{u : [0, \infty) \to U \mid u(.) \text{ measurable}\}\,.$$

For every $u(.)$ in $\mathcal{U}$ and every $p \in \mathbb{R}^n$, denote by $t \mapsto \gamma(p, u(.), t)$ the solution of (4) such that $\gamma(p, u(.), 0) = p$. Notice that, in general, $t \mapsto \gamma(p, u(.), t)$ needs not to be defined for every $t \geq 0$, since the non-autonomous vector field $f_{u(t)}$ may not be complete. Denote by $\mathcal{T}(p, u(.))$ the maximal element of $(0, +\infty]$ such that $t \mapsto \gamma(p, u(.), t)$ is defined on $[0, \mathcal{T}(p, u(.)))$, and let

$$Supp(\gamma(p, u(.), .)) = \gamma(p, u(.), [0, \mathcal{T}(p, u(.))))\,.$$

If $Supp(\gamma(p, u(.), .))$ is bounded, then $\mathcal{T}(p, u(.)) = +\infty$.

Given $p \in \mathbb{R}^n$, the *accessible set from* p, denoted by $\mathcal{A}(p)$, is defined as

$$\mathcal{A}(p) = \cup_{u(.) \in \mathcal{U}} Supp(\gamma(p, u(.), .))\,.$$

Several notions of stability for the switched system (4) can be introduced.

Definition 1 *We say that (4) is*

- **unbounded** *if there exist $p \in \mathbb{R}^n$ and $u(.) \in \mathcal{U}$ such that $\gamma(p, u(.), t)$ goes to infinity as t tends to $\mathcal{T}(p, u(.))$;*
- **bounded** *if, for every $K_1 \subset \mathbb{R}^n$ compact, there exists $K_2 \subset \mathbb{R}^n$ compact such that $\gamma(p, u(.), t) \in K_2$ for every $u(.) \in \mathcal{U}$, $t \geq 0$ and $p \in K_1$;*
- **uniformly stable** *at the origin if, for every $\delta > 0$, there exists $\varepsilon > 0$ such that $\mathcal{A}(p) \subset B_\delta$ for every $p \in B_\varepsilon$;*
- **locally attractive** *at the origin if there exists $\delta > 0$ such that, for every $u(.) \in \mathcal{U}$ and every $p \in B_\delta$, $\gamma(p, u(.), t)$ converges to the origin as t goes to infinity;*

- **globally attractive** *at the origin if, for every* $u(.) \in \mathcal{U}$ *and every* $p \in \mathbb{R}^n$, $\gamma(p, u(.), t)$ *converges to the origin as* t *goes to infinity;*
- **globally uniformly attractive** *at the origin if, for every* $\delta_1, \delta_2 > 0$, *there exists* $T > 0$ *such that* $\gamma(p, u(.), T) \in B_{\delta_1}$ *for every* $u(.) \in \mathcal{U}$ *and every* $p \in B_{\delta_2}$*;*
- **globally uniformly stable (GUS)** *at the origin if it is bounded and uniformly stable at the origin;*
- **locally asymptotically stable (LAS)** *at the origin if it is uniformly stable and locally attractive at the origin;*
- **globally asymptotically stable (GAS)** *at the origin if it is uniformly stable and globally attractive at the origin;*
- **globally uniformly asymptotically stable (GUAS)** *at the origin if it is uniformly stable and globally uniformly attractive at the origin.*

It has been showed by Angeli, Ingalls, Sontag, and Wang [4] that, when U is compact, the notions of GAS and GUAS are equivalent. This is the case for system (2). Moreover, it is well known that, in the case in which all the vector fields f_u are linear, local and global properties are equivalent.

2.2 The convexified system

In this paper, we focus on the planar switched system

$$\dot{q} = u\,X(q) + (1-u)\,Y(q)\,, \qquad q \in \mathbb{R}^2\,, \qquad u \in \{0,1\}\,, \tag{5}$$

where X and Y denote two vector fields on $\mathbb{R}^2$, of class $\mathcal{C}^\infty$, such that $X(0) = Y(0) = 0$. We assume moreover that X and Y are <u>globally asymptotically stable</u> at the origin. Notice, in particular, that X and Y are forward complete.

A classical tool in stability analysis is the convexification of the set of admissible velocities. Such transformation does not change the closure of the accessible sets. Moreover, it was proved in [10] (see also [4, Proposition 7.2]) that, for every $p' \in \mathbb{R}^2$, every switching function $u' : [0,\infty) \to [0,1]$, and every positive continuous function r defined on $[0, \mathcal{T}(p', u'(.)))$, there exist $u(.) \in \mathcal{U}$ and $p \in \mathbb{R}^2$ such that

$$\|\gamma(p, u(.), t) - \gamma(p', u'(.), t)\| \leq r(t)$$

for every $t \in [0, \mathcal{T}(p', u'(.)))$. As a consequence each of the notions introduced in Definition 1 holds for (5) if and only if it holds for the same system where $U = \{0,1\}$ is replaced by $[0,1]$.

In the following, to simplify proofs, we deal with the convexified system

$$\dot{q} = u\,X(q) + (1-u)\,Y(q)\,, \qquad q \in \mathbb{R}^2\,, \qquad u \in [0,1]\,. \tag{6}$$

Notations. When $u(.)$ is constantly equal to zero (respectively, one), we write $\gamma_Y(p,t)$ (respectively, $\gamma_X(p,t)$) for $\gamma(p, u(.), t)$. Given $p, p' \in \mathbb{R}^2$ and $u(.), u'(.)$ in $\mathcal{U}$, we say that $\gamma(p, u(.), .)$ and $\gamma(p', u'(.), .)$ *forwardly intersect* if $Supp(\gamma(p, u(.), .))$ and $Supp(\gamma(p', u'(.), .))$ have nonempty intersection.

2.3 The collinearity set of X and Y

A key object in order to detect stability properties of (6) turns out to be the set $\mathcal{Z}$ on which X and Y are parallel. We have that $\mathcal{Z} = Q^{-1}(0)$, where

$$Q(p) = det(X(p), Y(p))\,, \qquad p \in \mathbb{R}^2\,. \tag{7}$$

In [6], the stability of the linear switched system (3) was studied by associating with every point of $\mathbb{R}^2$ a suitably defined "worst" trajectory passing through it, whose construction was based upon $\mathcal{Z}$. The global asymptotic stability of the linear switched system (3) was then proved to be equivalent to the convergence to the origin of every such worst trajectory. We recall that in the linear case, excepted for some degenerate situations, $\mathcal{Z}$ is either equal to $\{0\}$ or is made of two straight lines passing through the origin.

In the nonlinear case, the situation is more complex. Let us represent $\mathcal{Z}$ as

$$\mathcal{Z} = \{0\} \cup \bigcup_{\Gamma \in \mathcal{G}} \Gamma, \tag{8}$$

where $\mathcal{G}$ is the set of all connected components of $\mathcal{Z} \setminus \{0\}$. Notice that $\mathcal{G}$ needs not, in general, to be countable. With a slight abuse of notation, we will refer to the elements of $\mathcal{G}$ as to the *components* of $\mathcal{Z}$.

Definition 2 *Let Γ be a component of $\mathcal{Z}$ and fix $p \in \Gamma$. We say that Γ is direct (respectively, inverse) if $X(p)$ and $Y(p)$ have the same (respectively, opposite) direction.*

Remark 3 *The definition is independent of the choice of p, since neither X nor Y vanish along Γ.*

An example of how $\mathcal{Z}$ can look like is represented in Figure 2.

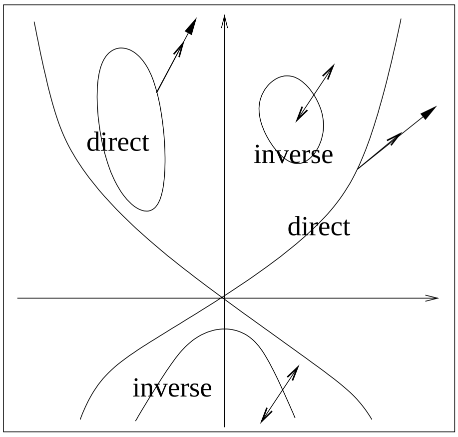

Figure 2: The set $\mathcal{Z}$

Some of the results of this paper are obtained assuming that the set $\mathcal{Z}$ has suitable regularity properties, which are generic in the sense defined below.

A base for the *Withney topology* on $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$ (the set of smooth vector fields on $\mathbb{R}^2$) can be defined, using the multi-index notation, as the family of sets of the type

$$\mathcal{V}(k, f, r) = \left\{ g \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2) \;\middle|\; \left\| \frac{\partial^{|I|}(f-g)}{\partial x^I}(x) \right\| < r(x), \forall x \in \mathbb{R}^2, |I| \le k \right\},$$

where k is a nonnegative integer, f belongs to $\mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R}^2)$, and r is a positive continuous function defined on $\mathbb{R}^2$. Denote by $\mathrm{GAS}(\mathbb{R}^2)$ the set of smooth vector fields on $\mathbb{R}^2$ which are globally asymptotically stable at the origin, and endow it with the topology induced by Withney's one. A *generic property* for (6) is a property which holds for an open dense subset of $\mathrm{GAS}(\mathbb{R}^2) \times \mathrm{GAS}(\mathbb{R}^2)$, endowed with the product topology of $\mathrm{GAS}(\mathbb{R}^2)$.

Lemma 4 *For a generic pair of vector fields (X, Y), $\mathcal{Z} \setminus \{0\}$ is an embedded one-dimensional submanifold of $\mathbb{R}^2$. Moreover, $Q(p)$ changes sign while p crosses $\mathcal{Z} \setminus \{0\}$.*

The lemma is a standard result in genericity theory. It follows from the fact that the condition

(G1) If $p \neq 0$ and $Q(p) = 0$, then $\nabla Q(p) \neq 0$, is generic (see, for instance, [1]). When $\mathcal{Z} \setminus \{0\}$ is a manifold, we say that $p \in \mathcal{Z} \setminus \{0\}$ is a tangency point if $X(p)$ is tangent to $\mathcal{Z}$. Under condition **(G1)**, $p \in \mathcal{Z} \setminus \{0\}$ is a tangency point if and only if $\nabla Q(p)$ and $X(p)$ (equivalently, $Y(p)$) are orthogonal.

Some of our results are obtained under additional generic conditions. One of these, namely,

(G2) The Hessian matrix of Q at the origin is non-degenerate,

ensures that $\mathcal{Z}$, in a neighborhood of the origin, is given either by $\{0\}$ or by the union of two transversal one-dimensional manifolds intersecting at the origin.

Under the generic conditions **(G1)** and **(G2)**, the connected component of $\mathcal{Z}$ containing the origin looks like one of Figure 3.

A third generic condition which we will sometimes assume to hold is

(G3) If $p \neq 0$, $Q(p) = 0$, and $\nabla Q(p)$ is orthogonal to $X(p)$, then the second derivative of Q at p along X (equivalently, Y) is different from zero,

which, together with **(G1)**, guarantees that the tangency points on $\mathcal{Z}$ are isolated.

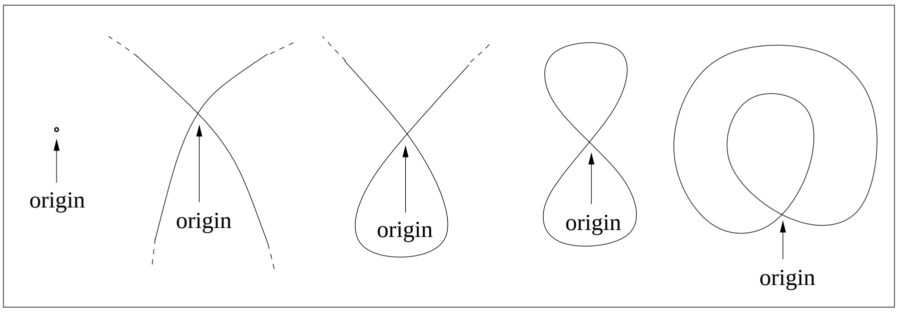

Figure 3: The connected component of $\mathcal{Z}$ containing the origin

3 Statement of the results

We organize our results in sufficient and necessary conditions with respect to the stability properties.

Notice that all such conditions are easily verified without any integration or construction of a Lyapunov function. Moreover, they are robust under small perturbations of the vector fields, as explained in Section 3.3. Let us recall that X and Y are assumed to be globally asymptotically stable at the origin and that all the results given below, although stated for the case $u \in [0,1]$, are also valid for the system where u varies in $\{0,1\}$.

Before stating our main theorems, observe that classical results on linearization imply the following.

Proposition 5 *Assume that the eigenvalues of $A = \nabla X|_{p=0}$ and $B = \nabla Y|_{p=0}$ have strictly negative real part. Then (6) is LAS if and only if (3) is GUAS.*

3.1 Sufficient conditions

The following theorem gives a simple sufficient condition for GUAS, which generalizes the analogous one already known for the linear system (3) (see [6, 14]).

Theorem 6 *Assume that $\mathcal{Z} = \{0\}$. Then the switched system (6) is GUAS at the origin.*

Under the generic assumptions **(G1)** and **(G2)**, Theorem 6 can be generalized as follows.

Theorem 7 *Assume that the generic conditions* **(G1)** *and* **(G2)** *hold. Assume, moreover, that the origin is isolated in $\mathcal{Z}$ and that there is no tangency point in $\mathcal{Z} \setminus \{0\}$. Then the switched system (6) is GUAS.*

When $\mathcal{Z}$ is bounded, although different from $\{0\}$, some weaker version of Theorem 6 still holds.

Theorem 8 *Assume that $\mathcal{Z}$ is compact. Then the switched system (6) is bounded.*

As a direct consequence of Proposition 5 and Theorem 8, we have the following sufficient condition for GUS.

Corollary 9 *Let $\mathcal{Z}$ be compact, and the linearized switched system be non-degenerate and GUAS. Then the switched system (6) is GUS.*

3.2 Necessary conditions

The following proposition expresses the straightforward remark that the inverse components of $\mathcal{Z}$ constitute obstructions to the stability of (6). The reason is clear: if Γ is inverse and p belongs to Γ, then a constant switching function $u(.)$ exists such that $\gamma(p, u(.), t) = p$ for every $t \geq 0$.

Proposition 10 *If $\mathcal{Z}$ has an inverse component, then the switched system (6) is not globally attractive.*

The following theorem gives a necessary condition for boundedness, under generic conditions.

Theorem 11 *Assume that the generic conditions* **(G1)** *and* **(G3)** *hold. If $\mathcal{Z}$ contains an unbounded inverse component, then the switched system (6) is unbounded.*

3.3 Robustness

We say that a property satisfied by (X, Y) is robust if it still holds for small perturbations of the pair (X, Y), that is, if it holds for all the elements of a neighborhood of (X, Y) in GAS($\mathbb{R}^2$)×GAS($\mathbb{R}^2$). Such notion of robustness is also known as *structural stability*, an expression which we prefer to avoid, in order to prevent confusion with the many definitions of stability already introduced for (6).

Under the generic conditions **(G1)** and **(G2)**, one can easily verify that the topology of the set $\mathcal{Z}$ does not change for small perturbations of X and Y. Moreover, fixed one component Γ of $\mathcal{Z}$, the fact that Γ is direct or inverse is robust. Similarly, if Γ is a component of $\mathcal{Z}$, which has not the origin in its closure, the absence of tangency points along Γ is robust. As a consequence, the conditions formulated by the theorems above are robust. More precisely:

Theorem 12 *Under generic assumptions, if any of Theorems 6, 7, 8, 11, Corollary 9, or Proposition 10 applies to the pair (X, Y), then it applies in a neighborhood of (X, Y) in GAS($\mathbb{R}^2$)×GAS($\mathbb{R}^2$).*

4 Proof of Theorem 6

Assume that $\mathcal{Z} = \{0\}$. We already recalled in Section 2 that GAS and GUAS are two equivalent notions. The main step of the proof consists in showing that (6) is globally attractive. The uniform stability will be obtained as a byproduct of the adopted demonstration technique.

Fix $q \in \mathbb{R}^2 \setminus \{0\}$. We first prove that $\mathcal{A}(q)$ is bounded. Then we show that, for every $u(.)$ in $\mathcal{U}$, the only possible accumulation point of $\gamma(q, u(.), t)$ is the origin. These two facts imply that $\gamma(q, u(.), t)$ converges to the origin as t goes to infinity.

4.1 Boundedness of $\mathcal{A}(q)$

We distinguish two cases.
First case: $\gamma_X(q, .)$ **and** $\gamma_Y(q, .)$ **do not forwardly intersect**. Then, we can define a closed, simple, piecewise smooth curve, by

$$\gamma_{X,Y}(q, t) = \begin{cases} \gamma_X(q, \tan(t\pi)) & \text{if } \ t \in \left[0, \frac{1}{2}\right], \\ \gamma_Y(q, \tan((1-t)\pi)) & \text{if } \ t \in \left[\frac{1}{2}, 1\right], \end{cases}$$

where $\gamma_X(q, \tan(\pi/2))$ and $\gamma_Y(q, \tan(\pi/2))$ are identified with the origin. The support of $\gamma_{X,Y}(q, .)$ separates $\mathbb{R}^2$ in two sets, one being bounded. Let us call $\mathcal{B}(q)$ the interior of the bounded set and $\mathcal{D}(q)$ the interior of the unbounded one.

Lemma 13 *$\mathcal{A}(q)$ is contained in $\overline{\mathcal{B}(q)} = \mathcal{B}(q) \cup \gamma_{X,Y}(q,[0,1])$.*

Proof. Consider the vector field $(X+Y)/2$. At the point q, it points either inside or outside $\mathcal{B}(q)$. Then, as it becomes clear through a local rectification of $(X+Y)/2$, the same holds true at all points of $\gamma_{X,Y}(q,[0,1])$ sufficiently close to q. Moreover, since the orientation defined by (X,Y) does not vary on $\mathbb{R}^2 \setminus \{0\}$ and coincides with the ones induced by $(X,(X+Y)/2)$ and $((X+Y)/2,Y)$, then $(X+Y)/2$ is pointing constantly either inside or outside $\mathcal{B}(q)$, all along $\gamma_{X,Y}(q,[0,1]) \setminus \{0\}$.

Let us assume that $(X+Y)/2$ points inside $\mathcal{B}(q)$. Then $\overline{\mathcal{B}(q)}$ is invariant for the flow of all the vector fields of the type $u\,X + (1-u)\,Y$, with $u \in [0,1]$, that is, it is invariant for the dynamics of (6). Hence, $\mathcal{A}(q)$ is contained in $\overline{\mathcal{B}(q)}$.

Assume now, by contradiction, that $(X+Y)/2$ points outside $\mathcal{B}(q)$. The same reasoning as above shows that $\mathcal{A}(q)$ is contained in $\overline{\mathcal{D}(q)}$. Define, for every $t \geq 0$ and every $\tau \in \mathbb{R}$,

$$\gamma^{X,Y,t}(q,\tau) = \begin{cases} \gamma_X(q,-\tau) & \text{if } \tau < -t\,, \\ \gamma_Y(\gamma_X(q,t),\tau+t) & \text{if } \tau > -t\,. \end{cases}$$

The support of $\gamma^{X,Y,t}$ is given by the union of the integral curves of X and Y connecting $\gamma_X(q,t)$ and the origin (see Figure 4). For every $t \geq 0$, we can identify $\gamma^{X,Y,t}$ with a closed curve passing through the origin.

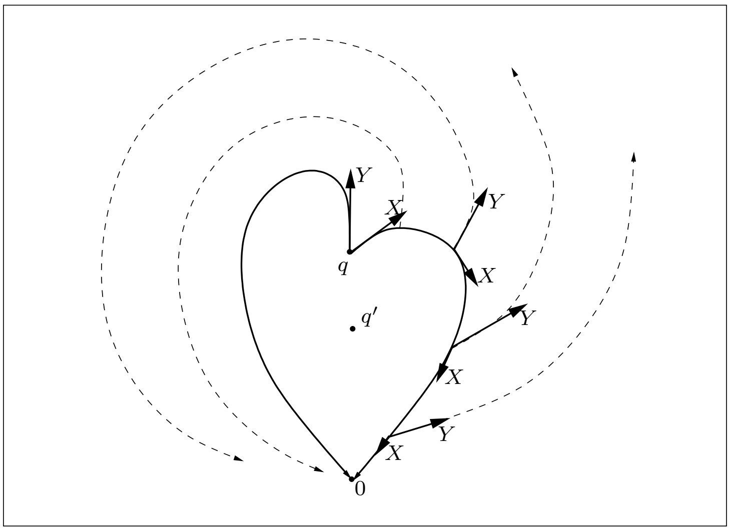

Figure 4: The curves $\gamma^{X,Y,t}$

Fix a point q' in $\mathcal{B}(q)$. By hypothesis, no $\gamma^{X,Y,t}$ passes through q'. Notice that the index of $\gamma^{X,Y,0}$ with respect to q' is equal to one, since the support of $\gamma^{X,Y,0}$ coincides with the boundary of $\mathcal{B}(q)$. The stability of Y at the origin implies that the index with respect to q' of the curve $\gamma^{X,Y,t}$ depends continuously of t, that is, it is constant on $[0,\infty)$. Hence, for every $t \in [0,\infty)$,

$$\max_{\tau\in\mathbb{R}} \|\gamma^{X,Y,t}(q,\tau)\| > \|q'\| > 0\,.$$

On the other hand, when t goes to infinity, $\gamma_X(q,t)$ converges to the origin and

$$\sup_{\tau<-t} \|\gamma^{X,Y,t}(q,\tau)\| = \sup_{\tau>t} \|\gamma_X(q,\tau)\| \xrightarrow{t\to\infty} 0\,.$$

Therefore, there exist $p \in \mathbb{R}^2$, arbitrarily close to the origin, such that the curve $s \mapsto \gamma_Y(p,s)$, $s > 0$, exits the ball $B_{\|q'\|}$, which contradicts the stability of Y at the origin. ■

<u>Second case:</u> $\gamma_X(q,.)$ **and** $\gamma_Y(q,.)$ **do forwardly intersect**. Let t be the first positive time such that the point $\gamma_Y(q,t)$ is equal to $\gamma_X(q,\tau)$ for some $\tau>0$. Define, for every $s\in[0,\tau+t]$,

$$\gamma_{X,Y}(q,s)=\begin{cases}\gamma_X(q,s) & \text{if} \quad s\in[0,\tau]\,,\\ \gamma_Y(q,t+\tau-s) & \text{if} \quad s\in[\tau,\tau+t]\,.\end{cases}$$

The curve $\gamma_{X,Y}(q,.)$ is simple and closed, and separates $\mathbb{R}^2$ in two open sets $\mathcal{B}(q)$ and $\mathcal{D}(q)$, $\mathcal{B}(q)$ being bounded.

Lemma 14 *$\mathcal{A}(q)$ is contained in $\overline{\mathcal{B}(q)}=\mathcal{B}(q)\cup\gamma_{X,Y}(q,[0,\tau+t])$.*

Proof. Assume that $(X+Y)/2$ points inside $\mathcal{B}(q)$ at q. Hence, the same is true for all points of $\gamma_{X,Y}(q,[0,\tau+t])$ sufficiently close to q. Since $\mathcal{Z}=\{0\}$, the property extends to the entire curve $\gamma_{X,Y}(q,[0,\tau+t])$, except possibly at the point $\gamma_{X,Y}(q,\tau)$. The same reasoning can be applied at $\gamma_{X,Y}(q,\tau)$, showing that $(X+Y)/2$ points either inside or outside $\mathcal{B}(q)$ at all points of the type $\gamma_{X,Y}(q,s)$, with s close to τ. The only non-contradictory possibility is that $(X+Y)/2$ points inside $\mathcal{B}(q)$ all along $\gamma_{X,Y}(q,[0,\tau+t])$. Hence, $\overline{\mathcal{B}(q)}$ is invariant under the flow of each vector field $u\,X+(1-u)\,Y$, $u\in[0,1]$, so that $\mathcal{A}(q)$ is contained in $\overline{\mathcal{B}(q)}$.

Assume, by contradiction, that $(X+Y)/2$ points inside $\mathcal{D}(q)$. The same reasoning as above shows that $\mathcal{A}(q)$ is contained in $\overline{\mathcal{D}(q)}$. In particular, the origin belongs to $\overline{\mathcal{D}(q)}$. On the other hand, the fact that $(X+Y)/2$ points outside $\mathcal{B}(q)$ all along $\partial\mathcal{B}(q)$ implies that it has a zero inside $\mathcal{B}(q)$. Which is impossible, unless $\mathcal{B}(q)$ contains the origin. ■

We proved that, in both cases, the set $\mathcal{A}(q)$ is bounded. The precise description of $\mathcal{A}(q)$ is given by the following lemma, where the definition of $\mathcal{B}(q)$ depends on whether $\gamma_X(q,.)$ and $\gamma_Y(q,.)$ forwardly intersect or not.

Lemma 15 $\mathcal{A}(q)=\overline{\mathcal{B}(q)}\setminus\{0\}$.

Proof. First notice that the origin does not belong to $\mathcal{A}(q)$, being a steady point for both X and Y. The inclusion of $\mathcal{A}(q)$ in $\overline{\mathcal{B}(q)}\setminus\{0\}$ is thus a consequence of Lemma 13 and Lemma 14.

As for the opposite inclusion, notice that $\partial\mathcal{B}(q)\setminus\{0\}$ is, by construction, made of integral curves of X and Y starting from q. Therefore, $\partial\mathcal{B}(q)\setminus\{0\}\subset\mathcal{A}(q)$.

Fix now $p\in\mathcal{B}(q)\setminus\{0\}$. We are left to prove that $p\in\mathcal{A}(q)$. Define

$$C=\{\gamma_X(p,\tau)\mid\tau\le 0\}\,,$$

and let V be a neighborhood of the origin such that $p\notin V$.

Due to the stability of X and the boundedness of $\mathcal{B}(q)$, there exists $T>0$ such that $\gamma_X(\overline{\mathcal{B}(q)},T)\subset V$. Since $\gamma_X(\gamma_X(p,-T),T)=p\notin V$, then C is not contained in $\overline{\mathcal{B}(q)}$. Therefore, there exists $\tau<0$ such that $\gamma_X(p,\tau)\in\partial\mathcal{B}(q)$. Notice that $\gamma_X(p,\tau)$ is different from the origin, since otherwise we would have $p=0$. Finally, $\gamma_X(p,\tau)\in\mathcal{A}(q)$, which implies that $p=\gamma_X(\gamma_X(p,\tau),|\tau|)$ belongs to $\mathcal{A}(q)$. ■

4.2 Global attractivity

In the previous section, we showed that the accessible set from every point is bounded. Hence, the global attractivity of (6) is proved if we ensure that no admissible curve has an accumulation point different from the origin.

Let us show that, for every point $p\neq 0$, there exist $\varepsilon>0$ and a neighborhood V_p of p such that every admissible curve $t\mapsto\gamma(q,u(.),t)$ entering V_p at time τ leaves V_p before time $\tau+\varepsilon$ and never comes back to V_p after time $\tau+\varepsilon$.

Since X and Y are not parallel at p, we can choose a coordinate system (x,y) such that $X(p)=(1,-1)$ and $Y(p)=(1,1)$. We denote $p=(p_x,p_y)$, $X(x,y)=(X_1(x,y),X_2(x,y))$, $Y(x,y)=(Y_1(x,y),Y_2(x,y))$. The fields X and Y being continuous, there exists $\alpha>0$ such that, if $(x,y)\in B_\infty(\alpha)=\{(a,b)\mid |a-p_x|<\alpha,\ \ |b-p_y|<\alpha\}$, then $X_1(x,y)$, $Y_1(x,y)$, $-X_2(x,y)$, and $Y_2(x,y)$ are in $[1/2,3/2]$.

Let $p'=(p_x-\frac{\alpha}{10},p_y)$ and consider $\gamma_X(p',.)=(\gamma_X^1(p',.),\gamma_X^2(p',.))$. Its first coordinate $\gamma_X^1(p',.)$ is increasing and its derivative takes values in $[1/2,3/2]$. The same is true for $-\gamma_X^2(p',.)$. Hence $\gamma_X(p',.)$ does not leave the set $B_\infty(\alpha)$ before time $2\alpha/3$. Since $\gamma_X^1(p', 2\alpha/5)$ is larger than $p_x + \frac{\alpha}{10}$ and $\gamma_X^2(p', 2\alpha/5)$ is in $[p_y - \frac{3\alpha}{10}, p_y - \frac{\alpha}{10}]$, then the curve $\gamma_X(p', .)$ intersects the segment $S_p = B_\infty(\alpha) \cap \{(x,y) | x = p_x + \frac{\alpha}{10}\}$ in a time τ_X smaller than $2\alpha/5$.

The same occurs for $\gamma_Y(p', .)$. Denote by τ_Y its intersection time with S_p.

Choose as V_p the bounded set whose boundary is given by the union of $\gamma_X(p', [0, \tau_X])$, $\gamma_Y(p', [0, \tau_Y])$, and the segment $[\gamma_X(p', \tau_X), \gamma_Y(p', \tau_Y)] = \{\lambda\, \gamma_X(p', \tau_X) + (1-\lambda)\, \gamma_Y(p', \tau_Y) \mid 0 \le \lambda \le 1\}$ (see Figure 5).

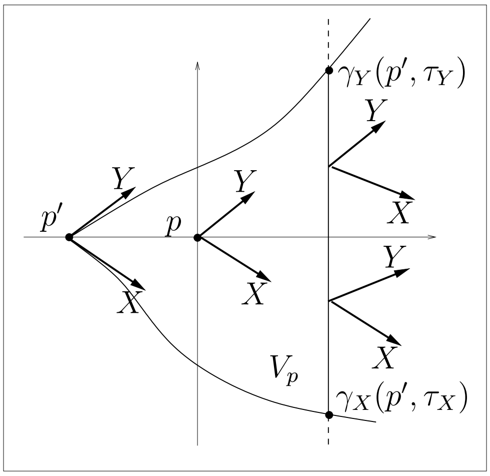

Figure 5: The set V_p

The following lemma states that V_p satisfies the required properties. As a consequence, p cannot be the accumulation point of any admissible curve.

Lemma 16 *We have the following: (i) V_p is a neighborhood of p; (ii) every admissible curve entering V_p leaves V_p in a time smaller than $2\alpha/5$ through the segment $[\gamma_X(p', \tau_X), \gamma_Y(p', \tau_Y)]$; (iii) once an admissible curve leaves V_p, it enters $\mathcal{A}(p') \setminus V_p$ and never leaves it.*

Proof. The first point follows by the construction of V_p. As for (*ii*), notice that all the points of V_p have first coordinate in $[p_x - \frac{\alpha}{10}, p_x + \frac{\alpha}{10}]$. Since the first coordinate of X and Y is larger than $1/2$, then every admissible curve entering V_p leaves it in a time smaller than $2\alpha/5$. Moreover, since along $\gamma_X(p', .)$ and $\gamma_Y(p', .)$ the admissible velocities of (6) point inside V_p, then an admissible curve can leave V_p only through the segment $[\gamma_X(p', \tau_X), \gamma_Y(p', \tau_Y)]$. Finally, (*iii*) follows from the remark that $\mathcal{A}(p') \setminus V_p$ is invariant for the dynamics, since the admissible velocities of (6) point inside $\mathcal{A}(p') \setminus V_p$ all along its boundary. ■

4.3 Conclusion of the proof of Theorem 6

We are left to prove that (6) is uniformly stable. To this extent, fix $\delta > 0$. Since both X and Y are stable at the origin, then there exists $\varepsilon > 0$ such that every integral curve of X or Y starting in B_ε is contained in B_δ. Hence, for every $q \in B_\varepsilon$, the boundary of $\mathcal{A}(q)$ is contained in B_δ. Therefore, $\mathcal{A}(q)$, being bounded, is itself contained in B_δ. ■

Remark 17 *The proof of Theorem 6 naturally extends to the following case: if V is an open and simply connected subset of $\mathbb{R}^2$, if X and Y point inside V along its boundary, and if $\mathcal{Z} \cap V = \{0\}$, then (6) is uniformly asymptotically stable on V.*

5 Proof of Theorem 7

The proof follows the main steps as the one of Theorem 6. The idea is again to fix a point $q \in \mathbb{R}^2$, to characterize the boundary of its accessible set $\mathcal{A}(q)$, to prove that such set is bounded, and, finally, to show that no admissible curve has an accumulation point different from the origin.

In order to describe the boundary of $\mathcal{A}(q)$, we need some extra construction. Notice that every component Γ of $\mathcal{Z}$ separates the plane in two parts. Since Γ contains no tangency points, then one of such two regions must be invariant for X, and the same argument holds for Y as well. Necessarily, the invariant region is the one containing the origin, which is attractive both for X and Y. In particular, Γ is direct and every admissible curve crosses Γ at most once. Associate with every point $q \in \mathbb{R}^2$ the number $n(q)$ of components of $\mathcal{Z}$ that the curve $\gamma_X(q,.)$ crosses at strictly positive times, before converging to the origin (see Figure 6). Since the curve $\gamma_X(q,(0,\infty))$ is bounded and crosses each component of $\mathcal{Z}$ at most once, then $n(q)$ is finite.

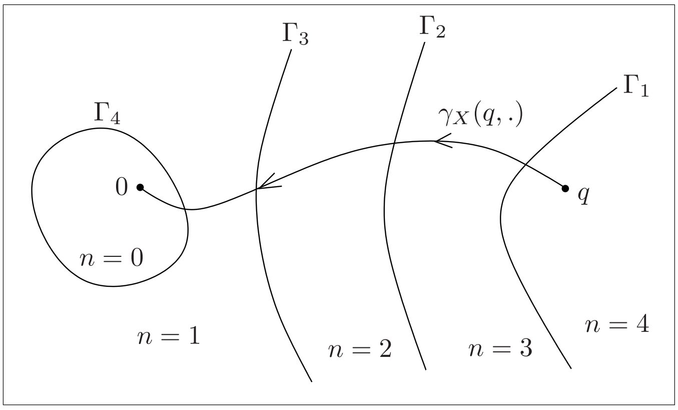

Figure 6

For every $i \leq n(q)$, let us denote by Γ_i the i-th component of $\mathcal{Z}$ crossed by $\gamma_X(q,.)$. We claim that $\gamma_Y(q,.)$ crosses exactly the same components as $\gamma_X(q,.)$, in the same order. Otherwise, as one can easily check, X and Y would not both be GAS at the origin (the reason is that the components of $\mathcal{Z}$ separate the plane and can be crossed by an admissible curve at most once).

Let us define two admissible curves, starting from q, that can be used to characterize the boundary of $\mathcal{A}(q)$, in analogy with what has been done in the proof of Theorem 6. The first of such curves follows the flow of X until it reaches Γ_1, then follows the flow of Y until it crosses Γ_2, and so on. The second one follows alternatively the flows of X and Y in the other way round, starting with Y and switching to X as it meets Γ_1. Such two curves converge to the origin, since $n(q)$ is finite. As in the proof of Theorem 6, we can distinguish two cases, depending on whether the two curves intersect or not (see Figure 7). The arguments of Section 4

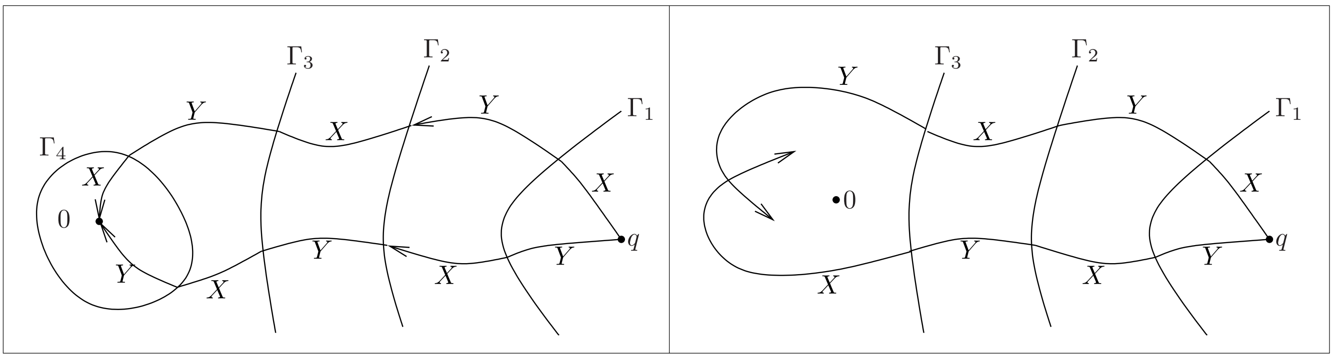

Figure 7

can be adapted in order to prove the boundedness of (6) and the absence of accumulation points different from the origin. The details are left to the reader. ■

6 Proof of Theorem 8

Consider a system of coordinates (x, y) on $\mathbb{R}^2$ which preserves the origin and renders X radial outside a ball B_{R_0}, $R_0 > 0$. (Such system can be defined using the level sets of a smooth Lyapunov function for X, see [9].) Taking possibly a larger R_0, we can assume that X and Y are never collinear in $\mathbb{R}^2 \setminus B_{R_0}$.

For every $R > 0$, let

$$\Omega_R = \cup_{p \in B_R} \mathcal{A}(p)\,.$$

Our aim is to prove that each Ω_R is bounded.

Fix $R > R_0 + 1$. If (X, Y) is replaced with a pair of vector fields (X', Y') which coincides with (X, Y) outside B_{R_0+1}, then the set Ω_R, constructed as above, does not change. The idea is to choose X' and Y' in such a way that they are never parallel outside the origin and still GAS. The boundedness of Ω_R follows then from Theorem 6.

Set

$$\begin{aligned} X_0(x, y) &= -x\partial_x - y\partial_y\,, \\ Y_0(x, y) &= y\partial_x - x\partial_y + \lambda X_0\,, \qquad \lambda > 0\,, \end{aligned}$$

and notice that X and X_0 are collinear outside B_{R_0}. Notice, moreover, that, if λ is large enough, then the angle between X_0 and Y_0 is smaller than the minimum of the angles between X and Y in $B_{R_0+1} \setminus B_{R_0}$ (see Figure 8). Fix such a λ.

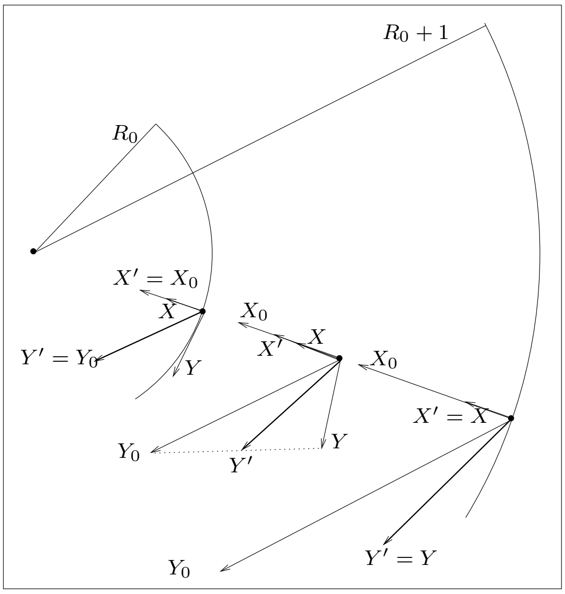

Figure 8

The function Q has constant sign on $\mathbb{R}^2 \setminus B_{R_0}$. Without lost of generality, we can assume that it is positive. Fix a smooth function $\phi : [0, +\infty) \to [0, 1]$ such that $\phi(r) = 0$ if $r \le R_0$ and $\phi(r) = 1$ if $r \ge R_0 + 1$. Define

$$\begin{aligned} X'(x, y) &= \left(1 - \phi\left(\sqrt{x^2 + y^2}\right)\right) X_0(x, y) + \phi\left(\sqrt{x^2 + y^2}\right) X(x, y)\,, \\ Y'(x, y) &= \left(1 - \phi\left(\sqrt{x^2 + y^2}\right)\right) Y_0(x, y) + \phi\left(\sqrt{x^2 + y^2}\right) Y(x, y)\,. \end{aligned}$$

By construction, (X', Y') coincides with (X, Y) outside B_{R_0+1} and $det(X', Y')$ is strictly positive on $\mathbb{R}^2 \setminus \{0\}$. We are left to check the global asymptotic stability of Y', the one of X' being evident. This can be done by using a comparison argument between the integral curves of Y and Y'. Indeed, since the angle between X' and Y' is smaller than the angle between X' and Y in $B_{R_0+1} \setminus B_{R_0}$, then the integral curve of Y' starting from a point $q \in \mathbb{R}^2 \setminus B_{R_0}$ joins B_{R_0} in finite time, with a smaller total variation in the angular component that the integral curve of Y starting from the same point q. ■

Remark 18 *The proof given above applies, without modifications, to the more general case where the points at which X and Y are globally asymptotically stable are allowed to be different.*

Remark 19 *The conclusion of Theorem 8 would not hold under the weaker hypothesis that X and Y are GUS, instead of GAS. A counterexample can be given as follows: Let $\varphi : [0, 1] \to \mathbb{R}$ be a smooth function such that $0 < \varphi(t) < \pi/2$ for every $t \in (0, 1)$ and $\varphi^{(k)}(0) = \varphi^{(k)}(1) = 0$ for every $k \geq 0$. Denote by (r, θ) the radial coordinates on $\mathbb{R}^2$. Define, using the radial representation of vectors in $\mathbb{R}^2$,*

$$X(r, \theta) = \begin{cases} \left(r, \theta + \frac{\pi}{2} + \varphi(r)\right) & \text{if } r \in [0, 1]\,, \\ \left(r, \theta + \frac{\pi}{2} - \varphi(r - [r])\right) & \text{if } r > 1\,, \end{cases}$$

and

$$Y(r, \theta) = \begin{cases} \left(r, \theta - \frac{\pi}{2} - \varphi(2r)\right) & \text{if } r \in \left[0, \frac{1}{2}\right]\,, \\ \left(r, \theta - \frac{\pi}{2} + \varphi\left(r + \frac{1}{2} - \left[r + \frac{1}{2}\right]\right)\right) & \text{if } r > \frac{1}{2}\,, \end{cases}$$

where $[r]$ denotes the integer part of r. Then, for every $r \geq 1$, $X(r, \theta)$ and $Y(r, \theta)$ are linearly independent, since the difference between their angular components is given by

$$0 < \pi - \varphi(r - [r]) - \varphi\left(r + \frac{1}{2} - \left[r + \frac{1}{2}\right]\right) < \pi\,.$$

Hence, $\mathcal{Z}$ is compact. On the other hand, the feedback strategy

$$u(t) = \begin{cases} 0 & \text{if } r - [r] \in \left[\frac{1}{4}, \frac{3}{4}\right)\,, \\ 1 & \text{otherwise} \end{cases}$$

is such that, for every $p \in \mathbb{R}^2 \setminus B_{3/4}$, $\|\gamma(p, u(.), t)\|$ tends to infinity as t tends to $\mathcal{T}(p, u(.)) = +\infty$.

Notice that the example can be easily modified in such a way that $\mathcal{Z}$ not only is compact, but actually shrinks to $\{0\}$. It suffices to take $X(r, \theta) = (r, \theta + \psi_X(r))$ and $Y(r, \theta) = (r, \theta + \psi_Y(r))$, where the graphs of ψ_X and ψ_Y are as in Figure 9.

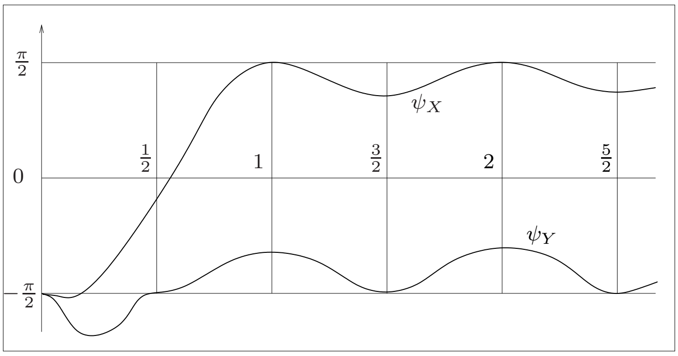

Figure 9

7 Proof of Theorem 11

Let Γ be an inverse unbounded component of $\mathcal{Z}$ and assume that **(G1)** and **(G3)** hold. Due to Lemma 4, Γ is a one-dimensional submanifold of $\mathbb{R}^2$, which can be parameterized by an injective and smooth map $c : \mathbb{R} \to \mathbb{R}^2$.

Fix a point $p = (p_x, p_y) = c(\tau)$ on Γ. According to the results by Davydov (see [7, Theorem 2.2]), up to a change of coordinates (which, in particular, sets $p_x = 0$), the vector fields X and Y can be represented locally by one of the following three normal forms

1. $X(x, y) = (1, x),\ Y(x, y) = (-1, x)$;
2. $X(x, y) = (1, y - p_y - x^2),\ Y(x, y) = (-1, y - p_y - x^2)$;
3. $X(x, y) = (-1, x^2 - y + p_y),\ Y(x, y) = (1, x^2 - y + p_y)$.

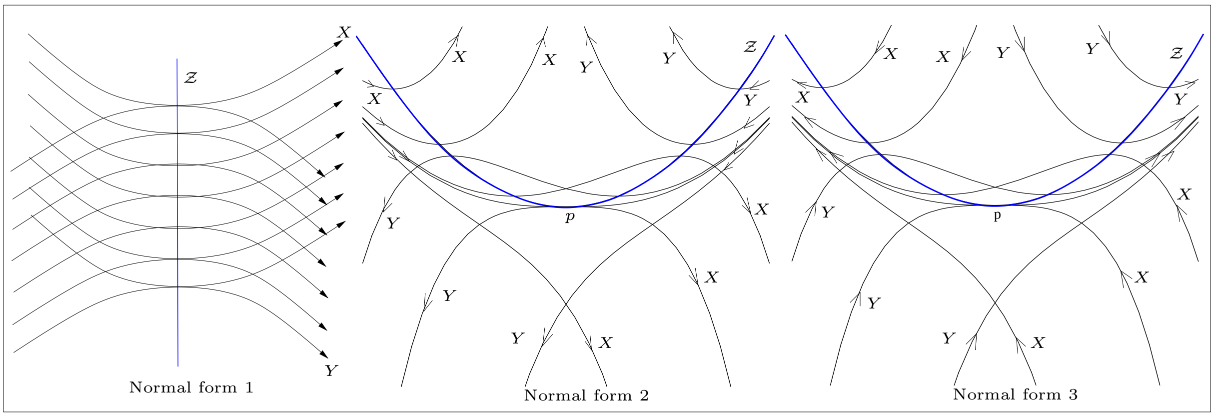

Figure 10

Notice that the type 1 corresponds to the situation in which X and Y are transversal to Γ at p, while 2 and 3 are the normal forms for the case in which X and Y are tangent to Γ at p.

Recall that p is said to have the *small time local transitivity* property (STLT, for short) if, for every $T > 0$ and every neighborhood V of p, there exists a neighborhood W of p such that every two points in W are accessible from each other within time T by an admissible trajectory contained in V. It has been proved in [7, Theorem 3.1] that, under the assumption that the system admits a local representation in normal form, p has the STLT property if and only if it is of the type 1. In particular, if p is of the type 1, then there exist $t(p), T(p) > 0$ such that, for every $r, s \in (-t(p), t(p))$ there exists an admissible trajectory which steers $c(\tau + r)$ to $c(\tau + s)$ within time $T(p)$.

Assume now that p is a point of the type 2 or 3. The curve Γ stays (locally) on one side of the affine line

$$p + \mathrm{span}(X(p)) = \{(x, p_y) \mid x \in \mathbb{R}\},$$

which is the affine tangent space to Γ at p. Up to a reversion in the parameterization of Γ, we can assume that, for every t in a right neighborhood of τ, $X(c(t))$ points into the locally convex part of the plane bounded by Γ (see Figure 10). It can be easily verified that the two branches of $\Gamma \setminus \{p\}$ are connected by integral curves of X and Y arbitrarily close to p, in the following sense: for every $t > 0$ small enough, there exist $\theta, T > 0$ such that, for every $r \in (0, \theta)$, both curves $s \mapsto \gamma_X(c(\tau + r), s)$ and $s \mapsto \gamma_Y(c(\tau - r), s)$ intersect Γ in a positive time smaller than T, and the intersection points are of the type $c(\tau + \rho)$, with $0 < |\rho| \le t$. We can conclude, using the STLT property at points of $\Gamma \setminus \{p\}$ close to p, that there exists $t(p) > 0$ such that, for every $\mu \in (0, 1)$, every two points of

$$\Sigma = \{c(\tau + r) \mid \mu\, t(p) < |r| < t(p)\}$$

can be joined by an admissible trajectory of time-length bounded by a uniform $T(p, \mu) > 0$.

Therefore, given any pair of points $p_i = c(\tau_i), p_f = c(\tau_f)$ on Γ of type 1, there exists an admissible trajectory going from p_i to p_f of time-length smaller that $T(c(\tau_1), \mu_1) + \cdots + T(c(\tau_k), \mu_k)$, where

$$(\tau_1 - t(c(\tau_1)), \tau_1 + t(c(\tau_1))), \ldots, (\tau_k - t(c(\tau_k)), \tau_k + t(c(\tau_k)))$$

is a covering of the compact segment of $\mathbb{R}$ bounded by τ_i and τ_f, $\mu_1, \ldots, \mu_k \in (0,1)$ are properly chosen and $T(p, \mu) = T(p)$ if p is of type 1. In particular, system (6) admits trajectories going to infinity. ■

Remark 20 *In the non-generic case the statement of Theorem 11 is false. A counterexample can be found even in the linear case. Indeed, consider the vector fields*

$$\begin{aligned} X(q) &= A\,q, \qquad \text{where} \quad A = \begin{pmatrix} -1/20 & -1/E \\ E & -1/20 \end{pmatrix}, \quad E = -\frac{201}{200} - \frac{\sqrt{401}}{200}, \\ Y(q) &= B\,q, \qquad \text{where} \quad B = \begin{pmatrix} -1/20 & -1 \\ 1 & -1/20 \end{pmatrix}. \end{aligned} \tag{9}$$

The integral curves of X are "elliptical spirals", while the integral curves of Y are "circular spirals". The integral curves of X and Y rotate around the origin in opposite sense (since $E < 0$). One can easily check that, in this case, the set $\mathcal{Z}$ is a single straight line of equation

$$y = -\frac{20}{\sqrt{401} - 1} x\,, \tag{10}$$

and its two components are inverse (see Figure 11).

It can be checked by hand that the switched system defined by X and Y is GUS, although not GUAS (see also [6], Theorem 2.3, case (CC.3)).

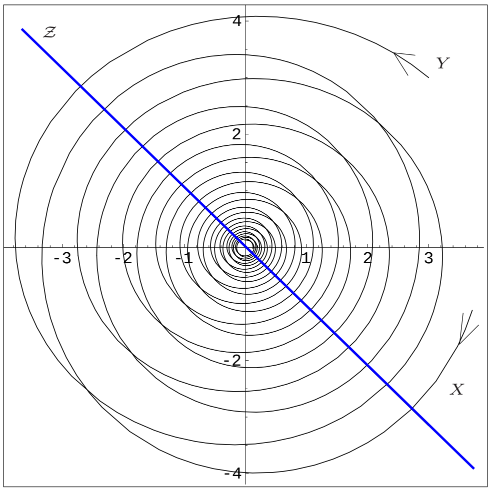

Figure 11

References

[1] R. Abraham and J. Robbin, *Transversal mappings and flows*, W. A. Benjamin, Inc., New York-Amsterdam, 1967.

[2] A. A. Agrachev and D. Liberzon, *Lie-algebraic stability criteria for switched systems*, SIAM J. Control Optim., **40** (2001), pp. 253–269.

[3] D. Angeli, *A note on stability of arbitrarily switched homogeneous systems*, to appear on Systems Control Lett.

[4] D. Angeli, B. Ingalls, E. D. Sontag, and Y. Wang, *Uniform global asymptotic stability of differential inclusions*, J. Dynam. Control Systems, **10** (2004), pp. 391–412.

[5] F. Blanchini and S. Miani, *A new class of universal Lyapunov functions for the control of uncertain linear systems*, IEEE Transactions on Automatic Control, **44** (1999), pp. 641–647.

[6] U. Boscain, *Stability of planar switched systems: the linear single input case*, SIAM J. Control Optim., **41** (2002), pp. 89–112.

[7] A. A. Davydov, *Qualitative theory of control systems*, Transl. Math. Monogr., Amer. Math. Soc., Providence, 1994.

[8] W. P. Dayawansa and C. F. Martin, *A converse Lyapunov theorem for a class of dynamical systems which undergo switching*, IEEE Trans. Automat. Control, **44** (1999), pp. 751–760.

[9] L. Grüne, E. D. Sontag, and F. R. Wirth, *Asymptotic stability equals exponential stability, and ISS equals finite energy gain—if you twist your eyes*, Systems Control Lett., **38** (1999), pp. 127–134.

[10] B. Ingalls, E. D. Sontag, and Y. Wang, *An infinite-time relaxation theorem for differential inclusions*, Proc. Amer. Math. Soc., **131** (2003), pp. 487–499.

[11] D. Liberzon, *Switching in systems and control*, Volume in series Systems & Control: Foundations & Applications, Birkhäuser, Boston, 2003.

[12] D. Liberzon, J. P. Hespanha, and A. S. Morse, *Stability of switched systems: a Lie-algebraic condition*, Systems Control Lett., **37** (1999), pp. 117–122.

[13] D. Liberzon and A. S. Morse, *Basic problems in stability and design of switched systems*, IEEE Control Syst. Mag., **19** (1999), pp. 59–70.

[14] P. Mason, U. Boscain, and Y. Chitour, *Common polynomial Lyapunov functions for linear switched systems*, Preprint SISSA 52/2004/M, `http://arxiv.org/abs/math.OC/0403209`.